Improved Lattice-Based Attack on Mersenne Low Hamming Ratio Search Problem

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Outline

1. Introduction
   1.1 Background
   1.2 Research Problem
2. Improved Attack
   2.1 Lattice
   2.2 Attack Strategy
   2.3 Success Probability
   2.4 Validating Experiments
3. Conclusion

1.1.1 BACKGROUND

AJPS Cryptosystem

- It was introduced by Aggarwal *et al.* at Crypto 2018
- A somewhat ring and noise scheme using elements of $\mathbb{Z}_p$
- Use Mersenne prime $p = 2^n - 1$ where n is a prime number

Interesting Features

- It is conjectured to resist against potential quantum attacks
- The advantage is simplicity in representation and computation
- Connection between integers modulo p and binary strings of length n

1.1.2 BACKGROUND

Key Information

- Integers in $\mathbb{Z}_p$ are mapped onto a set of n-bit strings
- Key generation involves random selection of f and g from $\mathbb{Z}_p$
- They relate to sparse binary strings with Hamming weight $w \approx \sqrt{n}$
- Another key h is defined as $f/g \pmod{p}$ ensuring g has an inverse
- It will relate to an n-bit string having an arbitrary Hamming weight

$$f,\ g \in \{0,\ 1,\ 2,\ \ldots,\ p-1\} \quad \rightleftharpoons \quad \underbrace{\cdots 0 \cdots \overbrace{1 \cdots 1 \cdots 1 \cdots 1}^{w \text{ ones}} \cdots 0 \cdots}_{n\text{-bit strings}}$$

Two Schemes

Single Bit Version

Two integers f and g each has a Hamming weight of w with a constraint $n > 4w^2$. The key pair $(\mathsf{pk}, \mathsf{sk})$ is $(h = f/g \pmod p),\ g)$.

Encryption

Choose a and b with a Hamming weight of w and encrypt one bit m through

$$c = (-1)^m \cdot (a \cdot h + b)$$

Decryption

Compute $d = \mathsf{Ham}(c \cdot g)$ and output '0' if $d \leq 2w^2$ or '1' otherwise. The core judgment is

$$c \cdot g = (-1)^m \cdot (a \cdot f + b \cdot g)$$

Two Schemes

Multiple Bits Version

Using above f, g and a random integer r modulo p leads to pk $:= (r, t) = (r, f \cdot r + g)$ and sk $:= f$. Besides, error correcting code $(\mathcal{E}, \mathcal{D})$ is required.

Encryption

Choose a, b_1, b_2 with a Hamming weight of w and encrypt multi-bit m to $(c_1, c_2) =$

$$(a \cdot r + b_1, (a \cdot t + b_2) \oplus \mathcal{E}(m))$$

Decryption

Output $\mathcal{D}((f \cdot c_1) \oplus c_2)$ as $f \cdot c_1$ and $a \cdot t + b_2$ exhibit a low Hamming distance through

$$f \cdot c_1 = (a \cdot t + b_2) - a \cdot g - b_2 + b_1 \cdot f$$

1.1.5 Hard Problems

Mersenne Low Hamming Ratio Search Problem (MLHRSP)

Consider an n-bit Mersenne prime $p = 2^n - 1$ and a positive integer w. Let f and g be two n-bit random strings characterized by a Hamming weight of w. The goal is to extract the values of f and g from the information provided by the equation $h = f/g \pmod p$ with a given h.

Mersenne Low Hamming Combination Search Problem (MLHCSP)

Consider an n-bit Mersenne prime $p = 2^n - 1$, a positive integer w, and a uniformly random n-bit string r. Let f and g be two n-bit random strings with a Hamming weight of w. The goal is to extract the values of f and g given $(r,\ t) = (r,\ f \cdot r + g \pmod p)$.

1.2.1 PREVIOUS ATTACKS

Beunardeau *et al.*'s Attack

When $f,\ g < \sqrt{p}$, $h = f/g \pmod{p}$ can be exploited to find them using a 2-dimensional lattice generated by basis matrix

$$\begin{pmatrix} 1 & h \\ 0 & p \end{pmatrix}.$$

Under Gaussian heuristic, recover a short vector (g, f) with 2^{-2w} probability.

More...

- A 3-dimensional lattice applies to the recovery of one bit m
- Similarly extending to attack on MLHCSP with $r, t = f \cdot r + g \pmod{p}$

1.2.2 Previous Attacks

Coron-Gini's Attack

It is a modified version of Beunardeau *et al.*'s attack on multi-bit AJPS and breaks the indistinguishability of ciphertexts (i.e., $m = 0$ and $m \neq 0$).

More...

- One has $\mathcal{E}(m) = 0$ for $m = 0$ and $c_1 = a \cdot r + b_1$, $c_2 = a \cdot t + b_2$
- Recovery of $a,\ b_1,\ b_2 < p^{2/3}$ through lattice reduction algorithm
- Success probability is $(2/3)^{3w} \approx 2^{-1.75w}$ outperforming original one

1.2.3 Our Contribution

Improved Lattice-Based Attack on MLHRSP

Let $p = 2^n - 1$ be an n-bit Mersenne prime and w be a positive integer. Let f and g bounded by $f \leq p^{\xi_1}$ and $g \leq p^{\xi_2}$, denote two unknown n-bit random strings with a Hamming weight of w. Given h satisfying $h = f/g \pmod p$, then f and g can be efficiently recovered if $\xi_1 + \xi_2 < 1$ (i.e., $f \cdot g < p$).

Improved Features

- Address unbalanced scenarios when $f < \sqrt{p} < g$ or $g < \sqrt{p} < f$
- Recognize unexplored advantage of lattice reduction algorithm
- Increase attack success probability from 2^{-2w} to $\sqrt{\pi}w^{3/2}/2 \times 2^{-2w}$

Lattice-Based Solving Strategy

Lattice Concepts

The set of all integer linear combinations of linearly independent vectors.

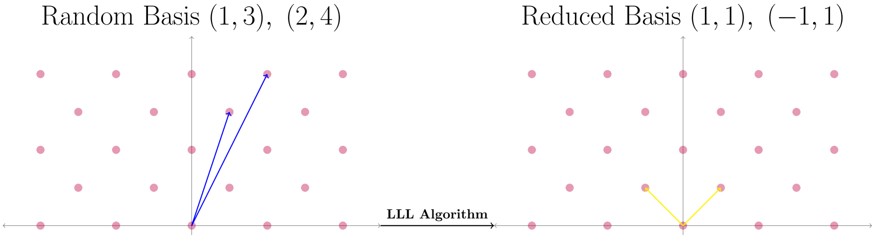
- Dimension: $\dim(\Lambda) = \omega$
- Basis vectors: $\vec{b}_1, \ldots, \vec{b}_\omega$
- Basis matrix: $B = (b_{ij})_{\omega \times \omega}$
- Determinant: $\det(\Lambda) = |\det(B)|$

$$\Lambda = \mathbb{Z}\vec{b}_1 + \cdots + \mathbb{Z}\vec{b}_\omega = \left\{ \sum_{i=1}^{\omega} z_i \vec{b}_i : z_i \in \mathbb{Z}, \vec{b}_i \in \mathbb{R}^\omega \right\}$$

Lattice-Based Solving Strategy

Lattice Reduction

- Lenstra, Lenstra, and Lovász proposed the famous LLL algorithm
- Output approximately shortest reduced vectors in polynomial time
- Lattice-based solving strategy is applied in public key cryptanalysis



2.1.3 Lattice-Based Solving Strategy

How to Find Small Modular Roots Using Lattice Reduction

1. Construct shift polynomials sharing common root modulo R
2. Transform their coefficient vectors into a lattice basis matrix B
3. Calculate short reduced vectors from ω-dimensional lattice $\Lambda(B)$
4. Transform output reduced vectors into integer equations system
5. Extract desired root over the integers using some simple methods

Asymptotic Solving Condition

$$\det(\Lambda) < R^{\omega} \quad \Longrightarrow \quad |\det(B)| < R^{\omega}$$

2.1.4 Target Modular Equation

Bivariate Equation

Derive a bivariate polynomial $f(x_1, x_2) := x_1 - hx_2$ from $h = f/g \pmod p$ and thus a bivariate modular equation:

$$f(x_1, x_2) \equiv 0 \pmod p,$$

with the root $(x_1^\star, x_2^\star) = (f, g)$. The upper bounds of desired root $(x_1^\star, x_2^\star)$ are $X_1 = p^{\xi_1}$ and $X_2 = p^{\xi_2}$ respectively.

Given parameters are as follows:

$$h, \quad p, \quad \xi_1, \quad \xi_2$$

2.2.1 Improved Strategy (1)

Shift Polynomials

Shift polynomials defined for a positive s and a non-negative i are

$$g_i(x_1, x_2) := x_2^{s-i} f^i(x_1, x_2) p^{s-i}, \quad 0 \le i \le s.$$

Therefore R indicated in the lattice-based solving strategy is p^s.

Improved Strategy

Shift Polynomials

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Therefore R indicated in the lattice-based solving strategy is p^s.

Coefficient Vectors

Transforming coefficient vectors of $g_i(X_1 x_1, X_2 x_2)$ into row vectors of B and the leading monomial of $g_i(x_1, x_2)$ is $x_1^i x_2^{s-i} p^{s-i}$.

Improved Strategy

Constructed Lattice

Regarding derived coefficient vectors as $\vec{b}_i$ for $i = 1, \ldots, \omega$ and generate

$$\Lambda = \left\{ \sum_{i=1}^{\omega} z_i \vec{b}_i : z_i \in \mathbb{Z} \right\}.$$

The lattice dimension ω is calculated as

$$\omega = \sum_{i=0}^{s} 1 = s + 1.$$

2.2.3 Improved Strategy (3)

Toy Example

$$
\begin{array}{c|ccc}
 & x_2^2 & x_1x_2 & x_1^2 \\
\hline
g_0 & p^2X_2^2 & 0 & 0 \\
g_1 & -hpX_2^2 & pX_1X_2 & 0 \\
g_2 & h^2X_2^2 & -2hX_1X_2 & X_1^2
\end{array}
$$

Improved Strategy

Toy Example

$$
\begin{array}{c|ccc}
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g_0 & p^2X_2^2 & 0 & 0 \\
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g_2 & h^2X_2^2 & -2hX_1X_2 & X_1^2
\end{array}
$$

Lattice Reduction

Matrix diagonals are $X_1^i X_2^{s-i} p^{s-i}$ for $0 \le i \le s$ and $\det(\Lambda) = p^{s_p} X_1^{s_1} X_2^{s_2}$ for $s_p = s_2 = \sum_{i=0}^{s}(s-i) = s(s+1)/2$ and $s_1 = \sum_{i=0}^{s} i = s(s+1)/2$.

Improved Strategy

Attack Bound

The solving condition $\det(\Lambda) < R^{\omega}$ with $R = p^s$ yields

$$(pX_1X_2)^{\frac{s(s+1)}{2}} < p^{s\cdot(s+1)}.$$

Simplify the exponents over p and obtain

$$\frac{1}{2} \cdot (1 + \xi_1 + \xi_2) < 1,$$

It further leads to

$$\boxed{\xi_1 + \xi_2 < 1, \quad (f \cdot g < p.)}$$

2.3.1 SUCCESS PROBABILITY

Previous Success Probability

Given f, g are both less than $\sqrt{p}$, namely their w many '1' bits are chosen from low $\lfloor n/2 \rfloor$ bits, the expression for Pr_1 is calculated as

$$\begin{aligned}\mathsf{Pr}_1 &= \frac{\binom{\lfloor n/2 \rfloor}{w}\binom{\lfloor n/2 \rfloor}{w}}{\binom{n}{w}\binom{n}{w}} \\ &= \left(\frac{\lfloor n/2 \rfloor!(n-w)!}{n!(\lfloor n/2 \rfloor - w)!}\right)^2 \\ &\approx 2^{-2w}.\end{aligned}$$

2.3.2 Success Probability

Our Success Probability

The w many '1' bits are chosen in a wider range and our Pr_2 is calculated as

$$\begin{aligned}
\mathrm{Pr}_2 &= \sum_{t=w}^{n-w} \frac{\binom{t}{w}\binom{n-t}{w}}{\binom{n}{w}\binom{n}{w}} = \frac{\binom{n+1}{2w+1}}{\binom{n}{w}\binom{n}{w}} \\
&= \frac{\binom{n+1}{2w+1}}{\binom{\lfloor n/2 \rfloor}{w}\binom{\lfloor n/2 \rfloor}{w}} \cdot \mathrm{Pr}_1 \\
&\approx \frac{\sqrt{\pi}}{2} w^{\frac{3}{2}} \cdot 2^{-2w} = \sqrt{\pi} w^{3/2} 2^{-2w-1}.
\end{aligned}$$

2.4. Experimental Results

Experiment Details

- Performed on a laptop computer running Ubuntu 22.04
- Conducted using `SageMath` mathematics software system
- Chose random parameters for generating experimental instances
- Provided source code at `https://github.com/MengceZheng/MLHRSP`

Time Comparison

- $n = 521, w = 10$: our improved attack succeeded in ≈ 0.2 s
- $n = 4253, w = 30$: our improved attack succeeded in ≈ 48 s
- $n = 11213, w = 50$: our improved attack succeeded in ≈ 2900 s

3. Conclusion

Improvements

- Expand vulnerable private key range and find more weak keys
- Increase success probability by considering unbalanced attack cases

Limitation

- Discard and resample f, g again if both of them fall within attack range

Future Work

- Explore how to incorporate a similar random partition technique
- Extend such improved lattice-based attack on MLHRSP to MLHCSP

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